to the problem under consideration we get

$$\frac{\sin \lambda x}{\sin x} = \frac{\lambda\left[1 - \left(\frac{\lambda x}{\pi}\right)^2\right]\left[1 - \left(\frac{\lambda x}{2\pi}\right)^2\right]\left[1 - \left(\frac{\lambda x}{3\pi}\right)^2\right]\cdots}{\left[1 - \left(\frac{x}{\pi}\right)^2\right]\left[1 - \left(\frac{x}{2\pi}\right)^2\right]\left[1 - \left(\frac{x}{3\pi}\right)^2\right]\cdots}.$$

Consider now the ratio of the nth binomial factor of the numerator to the nth or corresponding factor of the denominator:

$$\frac{1 - \left(\dfrac{\lambda x}{n\pi}\right)^2}{1 - \left(\dfrac{x}{n\pi}\right)^2} = \frac{n^2\pi^2 - \lambda^2 x^2}{n^2\pi^2 - x^2} = 1 + \frac{(1 - \lambda^2)x^2}{n^2\pi^2 - x^2}.$$

For any chosen value of n $(1, 2, 3, \cdots)$ the denominator of the last fraction decreases while the numerator increases as x grows larger. The fraction is always finite and positive because of the hypotheses $0 < \lambda < 1$ and $0 < x < \pi$. Consequently, the ratio of any binomial in the numerator of the expression for $(\sin \lambda x)/(\sin x)$ to the corresponding binomial in the denominator increases with x. It is accordingly manifest that the product of an infinite number of such converging ratios increases as x increases, and so the problem is solved.

II. Solution by Frank Irwin, University of California.

We shall show that, for the values of x in question, the derivative of the function is positive. This derivative is

$$\frac{\lambda \cdot \cos \lambda x \cdot \sin x - \sin \lambda x \cdot \cos x}{\sin^2 x}$$

or

$$\frac{1}{x \cdot \sin^3 x \cdot \sin \lambda x}(\lambda x \cdot \cot \lambda x - x \cdot \cot x).$$

This will be positive if $\lambda x \cdot \cot \lambda x > x \cdot \cot x$, that is, if $y \cdot \cot y$, let us say, continually decreases as y increases from 0 to π. This is so, since its derivative, $\cot y - y \csc^2 y$, or $(\sin 2y - 2y)/2 \sin^2 y$, is always negative.

A solution similar to the second was received from Elijah Swift.

392. Proposed by HORACE OLSON, Student at The University of Chicago.

Two right cylinders of radii a and b, respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

Solution by Elijah Swift, University of Vermont.

Assume $a > b$. Let the axis of the smaller cylinder be the z-axis, that of the larger, the y-axis. We see that the volume is

$$V = 8\int_0^b \int_0^{\sqrt{b-x^2}} \sqrt{a^2 - x^2} \cdot dy \cdot dx = 8\int_0^b \sqrt{(a^2 - x^2)(b^2 - x^2)} \cdot dx.$$

This latter integral is elliptic, and may be expressed in terms of complete elliptic integrals of the first and second kinds. Letting $x = b \operatorname{sn}(y, b/a)$, the integral becomes

$$8ab^2 \int_0^K \left\{ 1 - \frac{a^2 + b^2}{a^2} \operatorname{sn}^2 y + \frac{b^2}{a^2} \operatorname{sn}^4 y \right\} dy.$$

This, in turn, may be integrated by reduction formulas and gives finally for the volume,

$$V = \tfrac{8}{3}a\left[(b^2 - a^2)K\left(\frac{b}{a}\right) + (a^2 + b^2)E\left(\frac{\pi}{2}, \frac{b}{a}\right)\right],$$

where K denotes the complete elliptic integral of the first kind, E the elliptic integral of the second kind.

Note.—For this same problem, see Byerly's *Integral Calculus*, Chap. XVI, Elliptic Integrals. In the 1902 edition, however, an incorrect answer is given.

Also solved by H. S. UHLER.

MECHANICS.

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

SOLUTION BY H. S. UHLER, Yale University.

For brevity let

$$l \equiv \cos WOX = \cos XOW, \quad p \equiv \cos XOY = \cos YOX,$$
$$m \equiv \cos WOY = \cos YOW, \quad q \equiv \cos XOZ = \cos ZOX,$$
$$n \equiv \cos WOZ = \cos ZOW, \quad r \equiv \cos YOZ = \cos ZOY.$$

Since the forces are in equilibrium the (algebraic) sum of their projections on any straight line must vanish. Hence, by projecting the forces successively upon the lines of action of W, X, Y, and Z, respectively, we obtain the following redundant set of homogeneous equations:

$$W + lX + mY + nZ = 0,$$
$$lW + X + pY + qZ = 0,$$
$$mW + pX + Y + rZ = 0,$$
$$nW + qX + rY + Z = 0.$$

By applying a well-known theorem of determinants to the cofactors of the first row we find

$$W : X :: \begin{vmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: -\Delta_1^2 : \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}. \tag{1}$$

Similarly, for the second line or row

$$W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} 1 & m & n \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : -\Delta_2^2. \tag{2}$$

The determinants constituting the fourth and third terms of proportions (1) and (2), respectively, are equal because the rows of one are the same as the corresponding columns of the other, hence the product of (1) and (2) is

$$W^2 : X^2 :: \Delta_1^2 : \Delta_2^2,$$

or, since we are only dealing with the arithmetical magnitudes of the forces,

$$W : X :: \Delta_1 : \Delta_2.$$